Symmetries of near-horizon geometry equations

FACULTY OF PHYSICS UNIVERSITY OF WARSAW

Eryk Buk

Joint work with J. Lewandowski and A. Szereszewski

Abstract

We have calculated Lie point symmetries of near-horizon geometry equations. It turns out, that diffeomorphsms of independent variables are constrained. They must satisfy Cauchy-Riemann equations. Transformation of rotation one-form is composition of rescaling and rotation, similar to electromagnetic functions. Metric is rescaled and in the special case ($m = 0$) it can be interpreted as rescaling cosmological constant.

Near-horizon geometry

We consider horizon (null hypersurface) generated by Killing vector ℓ, positioned in electro-vacuum spacetime (greek indices). Examples of such hypersurface include horizons of stationary black holes. This vector satisfies:

$$\nabla_\mu \ell^\nu = \omega_\mu \ell^\nu \qquad a^{(\ell)} = \omega_\mu \ell^\mu, \tag{1}$$

where ω is called rotation one-form, and $a^{(\ell)}$ is surface gravity, dependent on choice of ℓ. We can choose Newman-Penrose null tetrad basis $(\ell, n, m, \bar{m})$, such that:

$$m^\mu \bar{m}_\mu = 1, \quad \ell^\mu n_\mu = -1. \tag{2}$$

For extremal Killing horizon ($a^{(\ell)} = 0$) of topology $\mathbb{R} \times S_2$ the following constraints are induced by Einstein-Maxwell equations:

$$\begin{cases} \nabla_{(A}\omega_{B)} - \omega_A \omega_B + \frac{1}{2} R_{AB} - \frac{1}{2}\Lambda g_{AB} - \kappa \left|\Phi_1\right|^2 g_{AB} = 0, \\ \left(\bar{\delta} + 2\pi\right)\Phi_1 = 0 \end{cases}, \tag{3}$$

where indices A, B pertain to local coordinates on S_2, and π is spin coefficient. These equations are called **near-horizon geometry equations** (NHG equations). Equivalent constraints are satisfied by horizons foliating Kundt's class spacetimes, extremal isolated horizons (by bijection) and Killing horizons.

Near-horizon geometry equations

Near-horizon geometry equations can be written as five scalar equations in local, real coordinates $x^A = (x^1, x^2)$ on S_2:

$$\begin{cases} 2\omega_{1,1} - (\omega_1\phi_{,1} - \omega_2\phi_{,2}) - 2(\omega_1)^2 - \frac{1}{2}(\phi_{,11} + \phi_{,22}) - \Lambda e^\phi \\ \qquad -2\kappa\left(U^2 + V^2\right)e^\phi = 0 \\ 2\omega_{2,2} + \omega_1\phi_{,1} - \omega_2\phi_{,2} - 2(\omega_2)^2 - \frac{1}{2}(\phi_{,11} + \phi_{,22}) - \Lambda e^\phi \\ \qquad -2\kappa\left(U^2 + V^2\right)e^\phi = 0 \\ \omega_{1,2} + \omega_{2,1} - (\omega_1\phi_{,2} + \omega_2\phi_{,1}) - 2\omega_1\omega_2 = 0 \\ U_{,1} - V_{,2} - 2(\omega_1 U - \omega_2 V) = 0 \\ U_{,2} + V_{,1} - 2(\omega_2 U + \omega_1 V) = 0 \end{cases} \tag{4}$$

We have decomposed Maxwell tensor's component Φ_1 into:

$$\Phi_1 = U + iV, \tag{5}$$

and used the fact that every two-dimensional metric is conformally flat to write metric on S_2 in the form:

$$g_{AB} = e^\phi \delta_{AB}, \quad \phi = \phi(x^1, x^2). \tag{6}$$

Prospects

One could try to find so called generalized symmetries, that is transformation whose generators are also functions of derivatives of dependent variables. Such procedure unfortunately is not algorithmic.

Lie point symmetries of differential equations

By symmetries of NHG equations (or any differential equation) we understand transformation of dependent and independent variables that leaves equations invariant (modulo this equation). To describe them we will be operating in second jet, which is locally a space of independent variables x^A, dependent variables u^α and all derivatives of dependent variables up to the second degree. In there, our NHG equations , which we mark as $H_k = 0$ ($k = 1, \ldots, 5$), defines a surface. Our aim is to find vector field X that preserves such surface and furthermore preserves relation between dependent variables and their derivatives, understood as local coordinates. This vector field generates diffeomorphisms of second jet which correspond to diffeomorphisms of dependent and independent variables generated by X_0 (whose prolongation is X). The vector field X associated with it is given by

$$\begin{aligned} X =& \xi^A(x^C, u^\beta)\frac{\partial}{\partial x^A} + \eta^\alpha(x^C, u^\beta)\frac{\partial}{\partial u^\alpha} + \\ & \eta^\alpha_A(x^C, u^\beta)\frac{\partial}{\partial u^\alpha_{,A}} + \eta^\alpha_{AB}(x^C, u^\beta)\frac{\partial}{\partial u^\alpha_{,AB}}, \end{aligned} \tag{7}$$

where η^α_A and η^α_{AB} can be expressed in terms of derivatives of functions $\xi^A(x^B, u^\beta)$ and $\eta^\alpha(x^A, u^\beta)$. Generators ξ^A and η^α do not depend on derivatives of u^α, which means our symmetries are called Lie point symmetries. The system of equations $H_k(x^A, u^\alpha) = 0$ must obey symmetry conditions:

$$X(H_k) = 0 \qquad \text{whenever} \qquad H_k = 0, \tag{8}$$

which gives us first order partial differential equations for both $\xi^A(x^B, u^\beta)$ and $\eta^\alpha(x^B, u^\beta)$. Once these functions are found, the flow generated by

$$X_0 = \xi^A(x^B, u^\beta)\frac{\partial}{\partial x^A} + \eta^\alpha(x^B, u^\beta)\frac{\partial}{\partial u^\alpha} \tag{9}$$

gives rise to the symmetries

$$\tilde{x}^A = \tilde{x}^A(x^B, u^\beta), \qquad \tilde{u}^\alpha = \tilde{u}^\alpha(x^B, u^\beta). \tag{10}$$

Lie point symmetries can be used to transform variables in a way that makes some of them redundant. It reduces differential equations to simpler form, making them easier to solve.

Results

Transformations of dependent and independent variables are given by:

$$\begin{aligned} \frac{d}{dt}\tilde{x}^1(t) &= \xi^1\left(\tilde{x}^1(t), \tilde{x}^2(t)\right) \\ \frac{d}{dt}\tilde{x}^2(t) &= \xi^2\left(\tilde{x}^1(t), \tilde{x}^2(t)\right) \\ \tilde{\omega}_1 &= \omega_1 e^{-\Xi_1}\cos(\Xi_2) + \omega_2 e^{-\Xi_1}\sin(\Xi_2) \\ \tilde{\omega}_2 &= -\omega_1 e^{-\Xi_1}\sin(\Xi_2) + \omega_2 e^{-\Xi_1}\cos(\Xi_2), \\ \tilde{\phi} &= \phi - 2\Xi_1 + m \\ \tilde{U} &= e^{-m/2}\left[U\cos(n_1 t) - V\sin(n_1 t)\right] \\ \tilde{V} &= e^{-m/2}\left[U\sin(n_1 t) + V\cos(n_1 t)\right] \end{aligned} \tag{11}$$

where m is constant (non-vanising only if $\Lambda = 0$), and the following equations are satisfied:

$$\begin{cases} \xi^1_{,1} = \xi^2_{,2} \\ \xi^1_{,2} = -\xi^2_{,1} \end{cases} \quad \text{and} \quad \begin{cases} \Xi_1 = \int_0^t \xi^1_{,1}(s)ds \\ \Xi_2 = \int_0^t \xi^1_{,2}(s)ds \end{cases}. \tag{12}$$

As can be seen from NHG equations, the case of $m \neq 0$ can be interpreted as rescaling Λ by e^m.

e-mail: ebuk@fuw.edu.pl